Received: *April 12, 2000,* Accepted: *May 17, 2000*

Self-duality, Ramond-Ramond fields and K-theory

Gregory Moore*

Department of Physics, Yale University, New Haven CT 06520, USA
E-mail: moore@castalia.physics.yale.edu

Edward Witten†

Department of Physics, Caltech, Pasadena, CA 91125, and
Caltech-USC Center for Theoretical Physics, Univ. of Southern California
E-mail: witten@ias.edu

JHEP05(2000)032

Abstract: Just as D-brane charge of type-IIA and type-IIB superstrings is classified, respectively, by $K^1(X)$ and $K(X)$, Ramond-Ramond fields in these theories are classified, respectively, by $K(X)$ and $K^1(X)$. By analyzing a recent proposal for how to interpret quantum self-duality of RR fields, we show that the Dirac quantization formula for the RR p-forms, when properly formulated, receives corrections that reflect curvature, lower brane charges, and an anomaly of D-brane world-volume fermions. The K-theory framework is important here, because the term involving the fermion anomaly cannot be naturally expressed in terms of cohomology and differential forms.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality.

*Address after January 1, 2000: Department of Physics, Rutgers University, Piscataway NJ 08855-0849 USA

†On leave from Institute for Advanced Study, Princeton NJ 08540

Contents

1. Introduction

The Ramond-Ramond (RR) fields of type-IIA superstring theory are differential forms $G_0, G_2, G_4, \ldots$ of all even orders, while for type-IIB superstring theory, one has RR fields $G_1, G_3, G_5, \ldots$ of all odd orders. The total RR field $G = G_0 + G_2 + G_4 + \cdots$ or $G = G_1 + G_3 + G_5 + \cdots$ is, classically, self-dual.

Self-duality alone introduces a number of subtleties in the study of the RR fields. For example, one expects a Dirac quantization condition for the G_p's, naively

$$\int_{U_p} \frac{G_p}{2\pi} \in \mathbb{Z}\,, \tag{1.1}$$

for every p-cycle U_p in spacetime. We will see that this statement receives modifications from several sources. Self-duality makes the interpretation of any such statement delicate, since classically one cannot impose the relation (1.1) for G_p and the dual field $G_{10-p} = *G_p$ at the same time; this point is explained in [1, section 3].

The effects of self-duality are most obvious for the five-form G_5 of type IIB. G_5 is self-dual, and this makes its dynamics particularly subtle, as has long been appreciated [2]. One approach to the quantum mechanics of such a field is to construct its quantum partition function by identifying the right theta function, as suggested in [3] and developed more explicitly in [4]. (References to a variety of other approaches to self-dual dynamics can be found in [1].) Except for this one case, one might hope at first sight to eliminate the subtleties of self-duality by eliminating the G_p of $p > 5$ using $G = *G$ and treating the G_p of $p < 5$ (or rather, their potentials) as the independent variables.

But things are not so simple. Eliminating the G_p's of $p > 5$ is unnatural since it is not invariant under T-duality. Moreover, the G_p's of different p are coupled in a subtle way by the reinterpretation [5, 6] of RR charges and (as we will argue) RR fields in K-theory. This again makes it subtle to eliminate half of them.

The goal of the present paper is to explain what statements along the line of (1.1) mean and what they say given the self-duality of RR fields and their interpretation in K-theory. For this we need first of all a precise framework for how to interpret RR fields (as opposed to charges) in K-theory. This is the subject of section 2. In brief, we propose that the cohomology class of the RR field of type-IIA superstring theory in a spacetime X is determined by an element of $K(X)$, while for type IIB it is determined by an element of $K^1(X)$. (The relation is stated in eq. (2.17) below.) This is precisely the opposite relation from the charges; we recall that RR charge of type IIA takes values in $K^1(X)$ (with a compact support condition), while for type IIB it takes values in $K(X)$. In section 3, we discuss how we implement self-duality in K-theory. In this, we follow the framework described in [1, sections 3 and 4.3] (which are recommended as background to the present paper); we summarize and amplify some key points. In particular, the mod 2 index of a certain Dirac operator plays an important role in this discussion. Finally in section 4, we apply this framework to the quantization conditions obeyed by the G_p's for type IIA.

For the special case of G_4, the shifted quantization law we get has been obtained before by considering global anomalies of membranes [7]. (That derivation was formulated for M-theory but applies equally in type IIA.) The shifts we get in general are natural from the standpoint of brane anomalies, even though we include no branes in the derivation. The shifted quantization laws cannot in general be naturally stated in cohomology, because there is in general no cohomological formula for the fermion global anomaly that enters in the analysis. This is another reason that it is important to describe RR fields via K-theory instead of cohomology.

Indeed, finding a natural framework in which to formulate the shifted quantization condition of G_6 was the original goal of the present paper. We started with the anomaly cancellation condition (4.4) and hoped to use it to understand just what kind of objects RR fields are. This proved difficult because it was hard to understand the role of the fermion anomaly. Following progress on understanding self-duality of RR fields in K-theory [1], it became clear, as we show in section 4, that in that framework the fermion anomaly term in the quantization condition comes in automatically. An important role in the K-theory framework is played by eq. (2.17), which we motivate in section 2 in a fairly elementary way using the same brane couplings that lead to eq. (4.4). This formula was suggested by D. Freed as an interpretation of eq. (4.4) with the fermion terms dropped.

In the present paper, the NS three-form field H is assumed to vanish. Including it raises a number of new issues, some of which will hopefully be addressed elsewhere [8]

2. RR fields and K-theory

Before considering type-II superstring theory, where the RR fields should really be interpreted in K-theory, let us consider the theory of an ordinary $(p-1)$-form potential C_{p-1}, with field strength G_p, in a $(d+1)$-dimensional spacetime X. (In superstring theory, $d=9$.) We will work first on a spacetime $X = \mathbb{R}\times M$, where $\mathbb{R}$ is the "time" direction and M is the spatial manifold. We assume that there are branes in M of codimension $p+1$ that serve as magnetic sources for C_{p-1}. (By replacing G_p with $*G_p$ in the following, we could similarly consider electric sources.) In the presence of such a brane with worldvolume W, G_p obeys

$$dG_p = \delta(W)\,, \tag{2.1}$$

where $\delta(W)$ is a $(p+1)$-form delta function that is Poincaré dual to W. Here we are writing this formula in the simple form that would hold for ordinary p-form fields (as opposed to the RR fields of type II, for which there are additional terms whose import is discussed below).

In general, for any brane worldvolume W in M, $\delta(W)$ is a closed $(p+1)$-form that defines an element $[W] \in H^{p+1}(M;\mathbb{Z})$. Thus, we interpret the brane charge in this situation as an element of this cohomology group. (2.1) says that $[W]$ is of the form $d(\cdots)$,[1] so that the cohomology class that represents the brane charge is zero (or else the equation for G_p has no solution).

If M is a compact manifold without boundary, this is the right answer: the total brane charge is identically zero. The way this is often stated is that the total charge associated with an abelian gauge symmetry vanishes on a compact manifold, since "the flux has nowhere to go".

For a setting in which the total brane charge is not zero, we consider the case that M is non compact with "boundary" N. We use the term "boundary" somewhat loosely; a typical case of interest is that $M = \mathbb{R}^d$, and N is the sphere S^{d-1} at infinity.

Even when M is not compact, (2.1) still implies that $[W]$ vanishes as an element of $H^{p+1}(M;\mathbb{Z})$. However, if we assume that W is compact, we can consider $[W]$ as an element of the compactly supported cohomology $H^{p+1}_{cpct}(M;\mathbb{Z})$. (2.1) says that the class of $[W]$ vanishes in $H^{p+1}(M;\mathbb{Z})$, but it does not imply that $[W]$ vanishes in $H^{p+1}_{cpct}(M;\mathbb{Z})$. The reason for this is that even if a G-field obeying (2.1) exists, it may not vanish at infinity. Vanishing of $[W]$ as an element of $H^{p+1}_{cpct}(M;\mathbb{Z})$ would imply that there exists a solution G of (2.1) that vanishes at infinity.

Thus, we should regard the brane charge as an element of $H^{p+1}_{cpct}(M;\mathbb{Z})$ that vanishes if mapped to $H^{p+1}(M;\mathbb{Z})$. The brane charge, in other words, takes values

[1] We assume that the eq. (2.1), which is stated for differential forms, is an approximation to an equation that holds for the integral cohomology.

JHEP05(2000)032

in the kernel of the natural map

$$i : H^{p+1}_{cpct}(M;\mathbb{Z}) \longrightarrow H^{p+1}(M;\mathbb{Z}) \tag{2.2}$$

which is defined by "forgetting" that a cohomology class has compact support.

So far we have tried to define the brane charge directly in terms of the brane worldvolume. However, it is often very useful in gauge theories to define the charge in terms of the behavior of the fields at infinity — or in this case, in terms of the restriction of G_p to N. Indeed, we have seen two paragraphs ago that the brane charge vanishes if G_p vanishes when restricted to N, strongly suggesting that the brane charge can be measured from the restriction of G_p to N.

We can get such a description of the brane charge using the long exact cohomology sequence for the pair (M, N). It reads

$$\cdots H^p(M;\mathbb{Z}) \xrightarrow{j} H^p(N;\mathbb{Z}) \longrightarrow H^{p+1}(M,N;\mathbb{Z}) \xrightarrow{i} H^{p+1}(M;\mathbb{Z}) \longrightarrow \cdots . \tag{2.3}$$

Here the relative cohomology $H^{p+1}(M,N;\mathbb{Z})$ is the same as the cohomology with compact support $H^{p+1}_{cpct}(M;\mathbb{Z})$. The map j is defined by restricting a cohomology class of M to the boundary N. From (2.3), we learn that

$$\ker(i) = \frac{H^p(N;\mathbb{Z})}{j\left(H^p(M;\mathbb{Z})\right)} . \tag{2.4}$$

This shows that the brane charge is determined by the cohomology class of the G-field in $H^p(N;\mathbb{Z})$, but that elements of $H^p(N;\mathbb{Z})$ that arise by restricting to N a cohomology class on M should be considered to represent zero brane charge. This has a simple intuitive interpretation. A G-field on N that extends over M as a closed p-form has no brane source and so has not been "created" by branes. Such a G-field is measurable on N but does not contribute to the brane charge, which takes values in the quotient indicated in (2.4).

Analog in K-theory. Now we move on to the type-II case, which differs in a few ways. There are G-fields of all even or all odd p, the brane W supports a Chan-Paton gauge bundle, and there are a number of subtle corrections to (2.1) involving lower brane charges [9, 10, 11, 5].

Thus, considering first type IIB, for branes of compact support in space, the brane charge is an element of $K_{cpct}(M)$, the compactly supported K-theory of M. To ensure that the equation for the RR fields has a solution, the brane charge must map to zero in $K(M)$. Thus, the brane charge takes values in the kernel of the natural map

$$i : K_{cpct}(M) \longrightarrow K(M) \tag{2.5}$$

which "forgets" that a K-theory class has compact support.

Just as in the case that the brane charge is interpreted as a cohomology class, we also want a description in which the brane charge is measured in terms of the RR fields at infinity. To see what form this must take, we look at the exact sequence that is the K-theory counterpart of (2.3). It reads

$$\cdots \longrightarrow K^{-1}(M) \xrightarrow{j} K^{-1}(N) \longrightarrow K(M,N) \xrightarrow{i} K(M)\cdots, \tag{2.6}$$

where again the relative K-group $K(M,N)$ is the same as $K_{cpct}(M)$, and j is the map that restricts a K-theory class on M to N. (This exact sequence has been used in computing $K_{cpct}(M)$ [12].) By the periodicity theorem, K^{-1} is the same as K^1. We see that the group $\ker(i)$ in which the brane charge takes values has an alternative description:

$$\ker(i) = \frac{K^1(N)}{j\left(K^1(M)\right)}. \tag{2.7}$$

We interpret this the same way that we did in the case of cohomology. $K^1(N)$ classifies RR fields at infinity, while $K^1(M)$ classifies RR fields on M that do not have any brane sources. An RR field on N that extends (as an element of K^1) over M does not require any brane sources, so the brane charges are classified by the quotient $K^1(N)/j(K^1(M))$.

This interpretation of (2.7) thus forces us to assert that type-IIB RR fields on M (or N) in the absence of branes are classified topologically by $K^1(M)$ or $K^1(N)$. This extends the relation of K-theory to RR charges that has been asserted in previous work.

Now we move on to the analogous situation for type IIA. Here, brane charge is classified by K^1. More specifically, a brane of compact support on M has a charge in $k^1_{cpct}(M)$, and (after requiring that the equation for the RR fields has a solution) the brane charge takes values in the kernel of

$$i : K^1_{cpct}(M) \longrightarrow K^1(M). \tag{2.8}$$

Once again, we can express the brane charge in terms of the fields at infinity. The exact sequence analogous to (2.6) is

$$\cdots \longrightarrow K(M) \xrightarrow{j} K(N) \longrightarrow K^1(M,N) \xrightarrow{i} K^1(M)\cdots, \tag{2.9}$$

where again $K^1(M,N)$ is the same as $K^1_{cpct}(M)$ and j is the restriction to N. Hence the brane charge takes values in

$$\ker(i) = \frac{K(N)}{j(K(M))}. \tag{2.10}$$

We interpret this to mean that for type IIA in the absence of branes, the RR fields on M (or N) are classified topologically by $K(M)$ (or $K(N)$).

Finally, for type I, D-brane charge is classified by $KO(M)$ with a compact support condition. Reasoning along the above lines leads to the conclusion that RR fields on $X = \mathbb{R} \times M$ are classified by $KO^{-1}(M)$.

The arguments we have given are slightly formal. But our conclusion that in the absence of branes the RR fields on a spacetime $X = \mathbb{R} \times M$ are classified in this way by K-theory seems to be the only reasonable way to reconcile the interpretation of brane charge in K-theory with the fact that in gauge theory, one expects to be able to measure the charges in terms of the fields at infinity. Since including the first factor in $X = \mathbb{R} \times M$ does not change the K-groups, we can equally well say that for X of the form $\mathbb{R} \times M$, the RR fields are classified (for type IIA, type IIB, and type I, respectively) by $K(X)$, $K^1(X)$, and $KO^{-1}(X)$.

Finally, we will take the additional leap of assuming that this result is not special to the case of $X = \mathbb{R} \times M$ that we have used to motivate the discussion. We will assume that (in the absence of branes) the RR fields on an arbitrary spacetime X, not necessarily of the form $\mathbb{R} \times M$, are classified topologically by $K(X)$ or $K^1(X)$ or $KO^{-1}(X)$.

Relation of $K(X)$ to RR fields. This last statement raises the following question, which we will consider first for type IIA. Given an element $x \in K(X)$ that determines an RR field G, what is the de Rham cohomology class of G?

To answer this question, we go back to the case that $X = \mathbb{R} \times M$. We consider a collection of 8-branes and $\overline{8}$-branes with world-volume $p \times M$, with p being a point in $\mathbb{R}$, and with arbitrary Chan-Paton bundles (E, F). Such configurations are classified topologically (modulo brane creation and annihilation [13, 14]) by the K-theory class x of the pair (E, F).

In crossing the branes, the de Rham cohomology class of the RR fields jumps. The jump is determined by the couplings of the RR fields to the brane. The relevant couplings were determined in [10]. They are usually expressed as electric couplings for the total RR potential $C = C_1 + C_3 + \cdots$. The couplings are

$$\int_{p \times M} C \wedge \sqrt{\widehat{A}} \Big(\mathrm{ch}(E) - \mathrm{ch}(F) \Big), \tag{2.11}$$

where ch is the Chern character. Here C_{2p-1} is the potential for the RR field G_{2p}. (An additional term is needed [15] to include the electric coupling of G_0, since there is no -1-form potential of G_0 in any standard sense; the effect of the addition is summarized in eq. (2.17) below.) A minus sign multiplies the second term of this expression because $\overline{8}$-branes have opposite sign couplings from 8-branes. Because of this minus sign and the fact that $E \to \mathrm{ch}(E)$ is a linear map from bundles to cohomology classes, it follows that the coupling on the right hand side in (2.11) depends only on the K-theory class x of the pair (E, F). We write the difference

$\mathrm{ch}(E) - \mathrm{ch}(F)$ more succinctly as $\mathrm{ch}(x)$, so we can rewrite (2.11):

$$\int_{p\times M} C \wedge \sqrt{\widehat{A}}\, \mathrm{ch}(x)\,. \tag{2.12}$$

In the presence of this coupling, the equation of motion of the RR field becomes

$$d(*G) = 2\pi\delta(p)\sqrt{\widehat{A}}\ \mathrm{ch}(x)\,, \tag{2.13}$$

where $\delta(p)$ is a delta function supported on $p \times M$. This equation implies that G jumps in crossing the brane. If we write G_L and G_R for the G-fields to the left and right of the brane, the jump is given by

$$*G_R - *G_L = 2\pi\sqrt{\widehat{A}}\ \mathrm{ch}(x)\,. \tag{2.14}$$

(Both X and M are oriented, so there is a natural left and right.)

Had we made a duality transformation, replacing G by $*G$, then (2.11) would be replaced by a magnetic coupling to the branes, which contributes to the Bianchi identity. With the magnetic coupling included, the Bianchi identity becomes

$$dG = 2\pi\delta(p)\sqrt{\widehat{A}}\ \mathrm{ch}(x)\,, \tag{2.15}$$

and this implies a jump in G of the form

$$G_R - G_L = 2\pi\sqrt{\widehat{A}}\ \mathrm{ch}(x)\,. \tag{2.16}$$

The magnetic coupling is more in the spirit of the derivation we gave above (in which branes were introduced as magnetic sources), and we will take (2.16) as the basic relation.

Of course, since G is supposed to be self-dual and the right-hand side of (2.14) or (2.16) is not self-dual, one should wonder what these equations mean. The most straightforward approach is to use self-duality to eliminate (for example) G_6, G_8, and G_{10}, treating the independent fields as G_0, G_2, and G_4. For G_0, G_2, and G_4 one would use (say) the magnetic coupling in (2.16), while the magnetic coupling of G_6, G_8, and G_{10} is included via an electric coupling of the potentials C_{2p-1}, $p \leq 2$, as in (2.11). In section 3, we will follow a more general formalism in which one can take an arbitrary "mutually commuting" set of RR periods as independent variables. In that framework, if M is a codimension 1 submanifold of a spacetime X, the restriction of a class in $K(X)$ to $K(M)$ is a set of commuting data that can be treated classically and is unconstrained by self-duality. In this framework, we can interpret (2.16) as a formula for the jumps of the cohomology classes of the G_p's for all p in crossing the brane.

Now we can finally make our proposal for the RR field determined by a K-theory class. Suppose that in this situation, the RR-fields are classified to the left of the branes by a K-theory class a, and to the right of the branes by a K-theory class

$b = a + x$. (2.16) expresses the difference between the G-field on the left and on the right. Let us assume that if $a = 0$ then G is zero (in de Rham cohomology) to the left of the brane. Then (2.16) determines G to the right of the brane, and we interpret this as the RR field of the K-theory class $b = x$:

$$\frac{G(x)}{2\pi} = \sqrt{\widehat{A}}\ \mathrm{ch}(x)\,. \tag{2.17}$$

We will assume that this formula holds for arbitrary spacetimes X in the absence of branes, and not just in the situation that we have used to motivate it.

We also need the type-IIB analog of (2.17). We propose that this is given by the same formula, but interpreted as follows. The RR fields of type IIB, in the absence of branes, are determined by an element $x \in K^1(X)$. $K^1(X)$ is the same as $\widetilde{K}(S^1 \times X)$ (the subgroup of $K(S^1 \times X)$ consisting of elements that are trivial if restricted to $q \times X$ for q a point in S^1). The Chern character $\mathrm{ch}(x)$ is hence an element of the even-dimensional cohomology of $S^1 \times X$. Upon integration over S^1, it maps to an element of the odd-dimensional cohomology of x. We will abbreviate this element as $\mathrm{ch}(x)$ (not showing the integration over S^1 in the notation). With this notational understanding, we propose (2.17) for both type IIA and type IIB. Given the type-IIA result, and assuming that the type-IIB formula should have the same general form, this is the unique formula for type IIB that is consistent with the requirement of T-duality between type IIA and type IIB in case $X = S^1 \times Y$ for some Y.

For type I, we again propose that the RR fields are determined for $x \in KO^{-1}(X)$ by the same formula (2.17), interpreted along the lines suggested in the last paragraph.

K-theory and unbroken symmetries. As well as classifying the RR fields, we also want to classify their symmetries. We begin again by recalling the case of ordinary p-form fields. A potential C_{p-1} with curvature G_p has a gauge-invariance $C_{p-1} \to C_{p-1} + dB_{p-2}$, with B_{p-2} a two-form gauge parameter. An unbroken gauge symmetry is a B_{p-2} such that $dB_{p-2} = 0$; they should be classified $\mathrm{mod}\, B_{p-2} \to B_{p-2} + da_{p-3}$ (for any $(p-3)$-form a_{p-3}) and modulo 2π shifts in the periods of B_{p-2}. The group of unbroken gauge symmetries is $H^{p-2}(X;\mathrm{U}(1))$. This group is not necessarily connected. By considering the long exact sequence in cohomology derived from the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 0\,, \tag{2.18}$$

one can show that its group of components $\overline{H}^{p-2}(X;\mathrm{U}(1))$ is the same as the torsion subgroup $H^{p-1}(X;\mathbb{Z})_{\mathrm{tors}}$ of $H^{p-1}(X;\mathbb{Z})$.

The analog for type IIA is that, while the RR fields are classified by $K(X)$, the unbroken gauge symmetries are classified by $K^{-2}(X;\mathrm{U}(1))$ (which by periodicity is the same as $K(X;\mathrm{U}(1))$), and its group of components is the torsion subgroup $K^1(X)_{\mathrm{tors}}$. In Fadde'ev-Popov gauge fixing, we will have to divide by the order of this group.

For type IIB, the group of components of the unbroken RR gauge symmetry group is $K(X)_{\mathrm{tors}}$, and for type I it is $KO^{-2}(X)_{\mathrm{tors}}$.

K-Theory and cohomology. $G/2\pi$ as determined in (2.17) is not an integral cohomology class, but (because of fractions in the power series expansion of $\sqrt{\widehat{A}}$ and ch) a rational one. We will interpret the RR fields G_p simply as p-forms, with no integral structure and no attempt to define the "torsion part" of G_p. The integral structure is defined at the level of K-theory, and in particular the torsion for RR fields of type IIA (in the absence of branes) is simply the torsion subgroup of $K(X)$.

It is illuminating to consider briefly some examples of how the passage from cohomology to K-theory mixes the RR forms. (The examples that follow are not used in the rest of the paper.) If for simplicity we consider a situation in which $\widehat{A}=1$ (getting rid of the most complicated fractions) and $G_0=G_2=0$, then (2.17) gives[2]

$$\frac{G_4}{2\pi}=c_2(x)\,,\qquad \frac{G_6}{2\pi}=\frac{c_3(x)}{2}\,. \tag{2.19}$$

Thus, in this situation, $G_4/2\pi$ is integral, but $G_6/2\pi$ is in general half-integral. Why is this half-integrality not seen in the simplest cases of brane physics? The most obvious case of quantization of G_6 arises in measuring the flux on an S^6 that links a D2-brane. (In this case, $\widehat{A}$, G_0, and G_2 are all zero or irrelevant, so our simplifying assumptions are valid.) Though $c_3(x)$ can be odd in general, it can be shown using the index theorem for the Dirac operator that c_3 is even for any complex vector bundle on S^6, so in this situation $G_6/2\pi$ is integral. In general, $c_3(x)$ is not even, but obeys a relation $c_3(x)\cong Sq^2(c_2(x)) \bmod 2$, where Sq^2 is a certain cohomology operation (a Steenrod square). In view of (2.19), this means that the half integral part of $G_6/2\pi$ is determined by G_4. This correlation between G_4 and G_6 is a typical illustration of the differences between K-theory and cohomology, and a fact that must be taken into account, along with the electric-magnetic duality between G_4 and G_6, in any detailed investigation of their properties. This and many other subtleties of the RR fields, which otherwise would have to be described piecemeal, are summarized by deriving the RR fields from K-theory.

For another illustration of the consequences of reinterpreting the RR fields in K-theory, we consider the torsion. There is no way in general to attribute elements of the torsion subgroup of $K(X)$, which we will call $K(X)_{\mathrm{tors}}$, to cohomology classes of X of a definite degree. For example, for $X=\mathbb{RP}^7$ (which arises in some orbifold

[2] In terms of the formal roots x_i of the Chern polynomial, $\mathrm{ch}(x)=\sum_i e^{x_i}$. As we assume $G_2=0$, we have $\sum_i x_i=0$. We also have $c_2=\sum_{i<j}x_ix_j$, $c_3=\sum_{i<j<k}x_ix_jx_k$, leading after some algebra to the following formulas.

studies and was considered in [12]), the even-dimensional cohomology of X is the sum $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where the summands are H^0, H^2, H^4, and H^6. However, $K(X) = \mathbb{Z} \oplus \mathbb{Z}_8$, where the summand $\mathbb{Z}$ corresponds to $H^0(X)$, but the summand $\mathbb{Z}_8$ is the K-theory analog of H^2, H^4, and H^6 combined. So, if RR fields are interpreted in K-theory, there is no way to make sense separately of the torsion part of G_2, G_4, and G_6. The generator of the $\mathbb{Z}_8$ factor of $K(\mathbb{RP}^7)$ is $x = \mathcal{L} - \mathcal{O}$, where $\mathcal{L}$ is a non-trivial flat line bundle over $\mathbb{RP}^7$ (it exists and is unique because $\pi_1(\mathbb{RP}^7) = \mathbb{Z}_2$) and $\mathcal{O}$ is a trivial line bundle. $c_1(x)$ is the non-trivial element of $H^2(\mathbb{RP}^7; \mathbb{Z}_2)$. The element $2\,x$ of $k(\mathbb{RP}^7)$ has $c_1(2\,x) = 0$, $c_2(2\,x) = c_1(x)^2 \neq 0$, proving that $2\,x$ is non zero in $k(\mathbb{RP}^7)$. The element $4\,x$ has all Chern classes zero, but is non-etheless non zero in $k(\mathbb{RP}^7)$. This example thus also shows that K-theory elements cannot always be classified by their Chern classes.

3. Self-duality, theta functions, and K-theory

3.1 Partition function on a closed manifold

Here we will describe how to interpret self-duality of quantum RR fields in the light of K-theory. To be more precise, in the limit (small string coupling or large volume) in which the RR fields can be treated as free fields, we will determine their quantum partition function on a compact manifold X, in the absence of branes. The discussion is largely a reprise of [1, section 4.3], repeated here to make this paper more readable, and with some extra details. For additional background about partition functions of self-dual p-forms for $p > 1$, the reader may consult [4] as well as [1, section 3]. We will carry out the discussion for type IIA, with brief comments later on type IIB and type I.

The first step is to introduce an anti-symmetric bilinear form $(\ ,\ )$ on $K(X)$. The definition is simply that (x, y) is the index of the Dirac operator on X with values in $x \otimes \overline{y}$. ($\overline{y}$ is obtained from y by complex conjugation of the bundles.) Since the dimension of X is of the form $4\,k + 2$, the index $i(w)$ of the Dirac operator with values in a K-theory class w obeys $i(w) = -i(\overline{w})$. Hence $(x, y) = -(y, x)$. Also, if x_0 is a torsion class, so that $n\,x_0 = 0$ for some integer n, then for any x,

$$(x, x_0) = \frac{1}{n}(x, nx_0) = \frac{1}{n}\,(x, 0) = 0\,. \tag{3.1}$$

Hence, if $K(X)_{\mathrm{tors}}$ is the torsion subgroup of $K(X)$, the bilinear form $(\ ,\ )$ is well defined as a bilinear form on the lattice $\Lambda = K(X)/K(X)_{\mathrm{tors}}$. It can be shown by imitating the proof of Poincaré duality given in [16] that the form $(\ ,\ )$ is unimodular on the lattice Λ.

The idea in quantizing the theory will be to write the partition function as a sum over a maximal "commuting" subgroup of $K(X)$. Here x and y are considered to commute if and only if $(x, y) = 0$. (One may suspect that one should somehow

JHEP05(2000)032

construct operators $\widehat{x}$ and $\widehat{y}$ that only commute under that condition.) In view of (3.1), every maximal commutative subgroup of $K(X)$ includes $K(X)_{\rm tors}$. We can always (albeit not in a unique or natural way) split $K(X)$ as $K(X) = \Lambda \oplus K(X)_{\rm tors}$. Given such a splitting, if Λ_1 is a maximal commutative sublattice of Λ, then a maximal commutative subgroup of $K(X)$ is $\overline{\Lambda}_1 = \Lambda_1 \oplus K(X)_{\rm tors}$. Every maximal commutative subgroup of $K(X)$ can be presented in this way. It is convenient to select a commuting sublattice Λ_2 of Λ that is complementary to Λ_1 (in the sense that $\Lambda = \Lambda_1 \oplus \Lambda_2$).

We also define a positive definite metric on Λ. It is defined by the formula

$$|x|^2 = \int_X G(x) \wedge *G(x) \,, \tag{3.2}$$

where $G(x)$ is a sum of differential forms G_p of all even p defined as follows. $G(x)$ is the unique *harmonic* differential form such that in de Rham cohomology, $G(x)$ is determined from x by the formula obtained in section 2:

$$\frac{G(x)}{2\pi} = \sqrt{\widehat{A}}\, {\rm ch}(x)\,. \tag{3.3}$$

This metric depends on the metric on X, and has no particular integrality properties.

Its attractive property is as follows. Let $\mathbb{T}$ be the torus $K(X;\mathbb{R})/\Lambda$, where $K(X;\mathbb{R}) = K(X) \otimes_{\mathbb{Z}} \mathbb{R}$. (Thus, $K(X;\mathbb{R})$ is isomorphic to $\mathbb{R}^n$, with $n = 2k$ the sum of the even Betti numbers of X.) Then the metric $|x|^2$ determines a metric g on $\mathbb{T}$, and the anti-symmetric form $(\ ,\ )$ determines a two-form ω on $\mathbb{T}$. The fact that $(\ ,\ )$ is integral and unimodular means that ω is integral and

$$\int_{\mathbb{T}} \frac{\omega^k}{k!} = 1\,. \tag{3.4}$$

ω is positive and of type (1,1) with respect to g, so together g and ω determine a Kahler structure on X.

The last ingredient one needs to set up the theory is a $\mathbb{Z}_2$-valued function Ω on $K(X)$ such that for all $x, y \in K(X)$,

$$\Omega(x+y) = \Omega(x)\,\Omega(y)(-1)^{(x,y)}\,. \tag{3.5}$$

A natural such function was defined in [1] as follows.[3] For x an element of *complex* K-theory, $x \otimes \overline{x}$ is naturally defined as an element of the *real* K-group $KO(X)$. Now we must use for the first time the fact that the spacetime dimension in string theory (namely 10) is of the form $8\,k+2$. (Our previous remarks are valid in any dimension

[3]A cocycle somewhat like Ω shows up in construction of vertex operator algebras from lattices, with the following difference. In that case, an Ω must be chosen, but the choice does not matter. Here, there is a distinguished Ω associated with the physical problem, and it is essential to find it.

of the form $4k + 2$.) In dimension $8k + 2$, there is a natural mod two function on $KO(X)$, namely the mod two index of the Dirac operator, which we will denote as j. (Concretely, if w is a real vector bundle — such as $x \otimes \overline{x}$ with x a complex vector bundle — then $j(w)$ is the number of positive chirality zero modes of the Dirac operator on X with values in w, mod 2. $j(w)$ is independent of the metric of X but in general depends on the spin structure. In [17], its dependence on the spin structure was expressed in terms of a relation similar to (3.5).) In [1], Ω was defined as

$$\Omega(x) = (-1)^{j(x \otimes \overline{x})} \tag{3.6}$$

and was shown to obey the basic identity (3.5).

(3.5) together with (3.1) implies that if x_0 is torsion, then

$$\Omega(x + x_0) = \Omega(x)\,\Omega(x_0)\,. \tag{3.7}$$

When we construct the partition function as a sum over a maximal commuting subgroup $\overline{\Lambda}_1$ of $K(X)$, $\Omega(x)$ will enter as a sign factor in the sum. All the factors in the partition function except $\Omega(x)$ are invariant under $x \to x + x_0$. Hence, given (3.7), the partition function will vanish under $x \to x + x_0$ unless Ω is identically 1 when restricted to $K(X)_{\mathrm{tors}}$. This vanishing cannot be removed by inserting local operators (as such operators do not receive contributions from the torsion), and must be interpreted as a kind of global anomaly, analogous to the anomaly discussed for M5-branes in [1, section 5.1].

We do not know whether there actually are ten-dimensional spin manifolds X such that Ω is non trivial on $K(X)_{\mathrm{tors}}$. If this does occur, the anomaly can be canceled as follows by wrapping a brane. (This paragraph is not essential in the rest of the paper.) On $K(X)_{\mathrm{tors}}$, Ω is multiplicative as in (3.7), and so is a homomorphism from $K(X)_{\mathrm{tors}}$ to $\mathbb{Z}_2 \subset \mathrm{U}(1)$. The effect of a K-theory class that is torsion is purely to include phases in the path integral for certain wrapped branes. (Some examples of this are discussed in detail in [18].) This suggests that the anomalous factor $(-1)^{h(x_0)}$ can be canceled by wrapping a brane. To prove that this is so, first recall from [1, section 5.1] the situation in cohomology. On a compact oriented d-manifold X, there is a Pontryagin duality

$$H^p(X;\mathbb{Z}) \times H^{d-p}(X;\mathrm{U}(1)) \longrightarrow \mathrm{U}(1)\,. \tag{3.8}$$

The pairing here is given by the cup product followed by integration; the fact that it is a Pontryagin duality can be proved by following the proof of Poincaré duality given in [16].[4] This duality induces a Pontryagin duality between the torsion

[4] One must modify the argument in [16] by replacing $\mathrm{Hom}(\ ,\mathbb{R})$ by $\mathrm{Hom}(\ ,\mathrm{U}(1))$; the argument goes through in the same way with this modification since $\mathrm{Hom}(\ ,\mathrm{U}(1))$, like $\mathrm{Hom}(\ ,\mathbb{R})$, maps exact sequences to exact sequences.

subgroup $H^p(X;\mathbb{Z})_{\rm tors}$ of $H^p(X;\mathbb{Z})$, and the group $\overline{H}^{d-p}(X;{\rm U}(1))$ of components of $H^{d-p}(X;{\rm U}(1))$:

$$H^p(X;\mathbb{Z})_{\rm tors}\times\overline{H}^{d-p}(X;{\rm U}(1))\longrightarrow{\rm U}(1)\,. \tag{3.9}$$

$\overline{H}^{d-p}(X;{\rm U}(1))$ is isomorphic (under the Bockstein $\beta: H^{d-p}(X;{\rm U}(1))\to H^{d-p+1}(X;\mathbb{Z})$) to $H^{d-p+1}(X;\mathbb{Z})_{\rm tors}$, so there is a Pontryagin duality

$$H^p(X;\mathbb{Z})_{\rm tors}\times H^{d-p+1}(X;\mathbb{Z})_{\rm tors}\longrightarrow{\rm U}(1)\,. \tag{3.10}$$

These Poincaré duality statements have analogs for K-theory.[5] On an oriented even-dimensional manifold X, the analog of (3.10) is the existence of a Pontryagin duality

$$K(X)_{\rm tors}\times K^1(X)_{\rm tors}\longrightarrow{\rm U}(1)\,. \tag{3.11}$$

The pairing here is just the physical coupling of a torsion RR field (an element of $K(X)_{\rm tors}$) to a torsion D-brane (which in type IIA determines an element of $K^1(X)_{\rm tors}$). This Pontryagin duality means that the homomorphism $\Omega: K(X)_{\rm tors}\to\mathbb{Z}_2\subset{\rm U}(1)$ is

$$x_0\longrightarrow(-1)^{(\alpha,x_0)}\,, \tag{3.12}$$

for some two-torsion element $\alpha\in K^1(X)_{\rm tors}$, where in (3.12) we write the pairing in an additive notation. Physically, this means that the anomaly will be canceled by wrapping a type-IIA D-brane that represents the class $\alpha\in K^1(X)$. In section 2, we argued that on a compact manifold X, the D-brane charge should vanish, but the anomaly means that actually the D-brane charge should equal the two-torsion element α. This is analogous to the situation in [19]: classical reasoning seems to show that the restriction of the class $[H]$ of the NS three-form field H to a D-brane world-volume Q should vanish, but because of an anomaly, $[H]$ should actually equal the two-torsion element $W_3(Q)$. The analogy is clear from [1, section 5.1], where the $W_3(Q)$ term shows up from the restriction of Ω to torsion.

For the rest of this paper, we restrict ourselves to the case that Ω is identically 1 when restricted to $K(X)_{\rm tors}$. In this case, the sum over the torsion subgroup will give a factor equal to the order of $K(X)_{\rm tors}$, which we will denote as N.

Moreover, if Ω is identically 1 on $K(X)_{\rm tors}$, then it can be regarded in a natural fashion as a function on the lattice $\Lambda=K(X)/K(X)_{\rm tors}$. A $\mathbb{Z}_2$-valued function on this lattice which obeys (3.5) determines (by a standard differential-geometric construction that was reviewed in [3]) a unitary line bundle $\mathcal{L}$ over $\mathbb{T}$ that has a

[5] Indeed, the main points in the proof of Poincaré duality in [16] are that there are Mayer-Vietoris sequences in cohomology theory and that Poincaré duality holds for $\mathbb{R}^d$. There are analogous Mayer-Vietoris sequences in K-theory, and the duality statements above are all true for $\mathbb{R}^d$. Of course, in making such a duality statement on $\mathbb{R}^d$ (or in general on a non-compact oriented manifold) one must understand one of the two factors, such as H^p or H^{d-p} in (3.8), to be cohomology with compact support.

JHEP05(2000)032

connection with curvature form ω. As ω is of type $(1,1)$, $\mathcal{L}$ has a natural holomorphic structure. (3.4) together with the Riemann-Roch theorem implies that $\mathcal{L}$ has a unique holomorphic section Θ.

For any decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ of Λ in terms of commutative sublattices Λ_1 and Λ_2, Λ can be written as a sum over certain cosets of Λ_1. This was explained in [1, section 3]. To be more precise, because of the duality between Λ_1 and Λ_2, there exists $\theta \in \Lambda_1$ such that, for $y \in \Lambda_2$, $\Omega(y) = (-1)^{(\theta,y)}$. The theta function is

$$\Theta = \sum_{x \in \Lambda_1 + \frac{1}{2}\theta} \Omega(x - \theta/2) \exp\left(i\pi(x, \tau x)\right), \tag{3.13}$$

where τ is the period matrix of the lattice Λ (with respect to its decomposition as $\Lambda_1 \oplus \Lambda_2$). For the analogous theory of self-dual p-forms, a fairly explicit explanation of how the period matrix comes in is in [4].

Once the theta function is constructed, the partition function of the RR fields (assuming that $\Omega = 1$ for torsion elements) is

$$Z = \frac{\Theta}{\Delta}. \tag{3.14}$$

Here Δ is a determinant of the non-zero modes and is completely unaffected by all of the subtleties that we have discussed. Δ would be the same if RR fields were ordinary differential forms, not related to K-theory. Δ has been treated for a self-dual p form in [4, section 4]; for a self-dual three-form on $\mathbb{T}^6$, it has been explicitly incorporated in the computation in [20].

One might expect in the numerator in (3.14) a factor of N from summing over $K(X)_{\mathrm{tors}}$, but this factor is canceled in the following way. In Fadde'ev-Popov gauge fixing, one must divide by the volume of the unbroken gauge symmetry group, which is proportional to the number of components of that group. As we have argued in section 2, the group of components of the unbroken RR gauge symmetry for type IIA is $K^1(X)_{\mathrm{tors}}$. The existence of the perfect pairing (3.11) means that the order of $K^1(X)_{\mathrm{tors}}$ is the same as the order of $K(X)_{\mathrm{tors}}$, so the factor in the numerator that comes from summing over torsion is canceled by a similar factor in the denominator. This cancellation is invariant under T-duality to type IIB, which exchanges the roles of $K^1(X)_{\mathrm{tors}}$ and $K(X)_{\mathrm{tors}}$!

The formula (3.13) shows that in writing the partition function as a sum over a maximal commuting sublattice of Λ, one in general has to sum over, roughly speaking, half-integral elements of $K(X)$. If one chooses to express the partition function as a sum over certain RR fields via (3.3), the conditions on the RR fields that must be included in the sum are much more complicated.

For type IIB, one repeats this analysis, using $K^1(X)$ instead of $K(X)$; the definitions of $(\ ,\ )$ and of Ω are given in [1]. The analogous construction for type I is as follows. First we have to define an anti-symmetric bilinear form $(\ ,\ )$ on

$KO^{-1}(X)$. We recall that $KO^{-1}(X) = \widehat{KO}(S^1 \times X)$, where $\widehat{KO}(S^1 \times X)$ is the subgroup of $KO(S^1 \times X)$ consisting of elements that are trivial if restricted to $p \times X$ for p a point in S^1. For $x, y \in \widehat{KO}(S^1 \times X)$, we have $x \otimes y \in \widehat{KO}(S^1 \times S^1 \times X)$. We define (x, y) to be one half the index of the Dirac operator on $S^1 \times S^1 \times X$ with values in $x \otimes y$; this is an integer, as $x \otimes y$ is real and $S^1 \times S^1 \times X$ has dimension of the form $8k + 4$. As for $\Omega(x)$, we reason as in the discussion of type IIB in [1, section 4.3]. $x \otimes x$ can be interpreted as an element of $KR(S^2 \times X)$, where the involution used in defining KR is a reflection of one coordinate on S^2. By the periodicity theorem, $KR(S^2 \times X) = KO(X)$, and we define $\Omega(x) = (-1)^{j(\pi(x))}$, where $\pi(x)$ is the image of $x \otimes x$ in $KO(X)$ and $j(\pi(x))$ is its mod two index.

3.2 Extension to a manifold with boundary

Now we will make an important extension that goes beyond what has been said in [1]. The goal is to show, at least in part, that the formalism we have sketched above respects the locality of quantum field theory. (In what follows, we make use of a mathematical result along the lines of [21]. See [22] for the roughly analogous but more subtle case of the gluing behavior of the eta invariant.)

Using the mod 2 index, we have defined the factor $\Omega(x)$ on a closed manifold X. We want to extend the definition to define $\Omega(x)$ on a manifold with boundary, in such a way that if X_1 and X_2 have a common boundary component B (with opposite orientations), and X is obtained by gluing together X_1 and X_2 along B, then Ω will be multiplicative in the gluing. This multiplicativity means that for $x \in K(X)$, if x_1 and x_2 are the restrictions of x to X_1 and X_2, then

$$\Omega(x) = \Omega(x_1)\, \Omega(x_2)\,. \tag{3.15}$$

(One can also formulate a similar gluing law for the case that X_1 has two boundary components both isomorphic to Y, and X is obtained by gluing them together.) However, for a manifold X_1 with boundary, $\Omega(x_1)$ will not be defined simply as an element of the group $\mathbb{Z}_2$. It will be defined as an element of a non-trivial principal $\mathbb{Z}_2$ bundle $\mathcal{P}$.

In general, let X be a ten-dimensional spin manifold with a boundary Y, of dimension nine. (We do not assume that Y is connected.) If we write y for the restriction of x to Y, then $y \otimes \overline{y}$ is a real vector bundle. In dimension $8k + 1$, the gamma matrices are real, and the Dirac operator $\mathcal{D}_Y = \sum_i \Gamma^i D_i$ with values in the real bundle $y \otimes \overline{y}$ is a real anti-symmetric (or anti-hermitean) operator. Such an operator in general has a mod 2 index, but the mod 2 index of $\mathcal{D}_Y$ vanishes, since the mod 2 index is a bordism invariant, and Y is the boundary of the spin manifold X, over which $y \otimes \overline{y}$ extends. We will only consider the case that the mod 2 index is zero on each component of Y; some additional subtlety is involved in extending the discussion when this is not true. Under this hypothesis, for a generic metric on

Y and connection on y, $\mathcal{D}_Y$ has no zero eigenvalues, and the fermion path integral for fermions on Y with values in $y \otimes \overline{y}$ is non zero. This path integral, which is often denoted as $\sqrt{\det \mathcal{D}_Y}$, is most naturally understood as the Pfaffian of the real anti-symmetric operator $\mathcal{D}_Y$, so we will write it as $\mathrm{Pf}(\mathcal{D}_Y)$. This Pfaffian is subject to an anomaly. The absolute value $|\,\mathrm{Pf}(\mathcal{D}_Y)|$ is naturally defined as a real number (for example, using zeta function regularization), but there can be an anomaly in the sign of the Pfaffian. The most natural way to describe mathematically this sign anomaly is to say that $\mathrm{Pf}(\mathcal{D}_Y)$ is not a real number, but takes values in a real line bundle, called the Pfaffian line bundle. We will denote this line bundle as $\mathrm{Pf}(Y)$. The structure group of the Pfaffian line bundle $\mathrm{Pf}(Y)$ is the subgroup $\{\pm 1\}$ of the real numbers (this is just a fancy way to say that $\mathrm{Pf}(\mathcal{D}_Y)$ is well-defined up to sign); this group is isomorphic to $\mathbb{Z}_2$. So we can build a principal $\mathbb{Z}_2$ bundle $\mathcal{P}(Y)$ over the parameter space (of metrics and gauge fields on Y) using the same structure functions as those of $\mathrm{Pf}(Y)$. For $x \in K(X)$, we will define $\Omega(x)$ as a section of $\mathcal{P}(Y)$. Both $\mathrm{Pf}(Y)$ and $\mathcal{P}(Y)$ depend on y, but we do not show this in the notation. A fancy way to express the relation between them is that $\mathrm{Pf}(Y) = \mathcal{P}(Y) \otimes_{\mathbb{Z}_2} \eta$, with η a trivial real line bundle on which $\mathbb{Z}_2$ acts as the group $\{\pm 1\}$.

To facilitate the later discussion, we make a few observations about the Dirac operator on Y. The hermitean operator $i\mathcal{D}_Y$ has real eigenvalues, found by solving the eigenvalue problem

$$i\mathcal{D}_Y \psi = \lambda \psi, \quad \lambda \in \mathbb{R}. \tag{3.16}$$

Let V be the space of all real spinor fields on Y with values in $y \otimes \overline{y}$, and let $V_{\mathbf{C}}$ be the complexification of V. On V there is a positive definite metric

$$\langle \psi_1, \psi_2 \rangle = \int_Y d^9x \sqrt{g}\, (\psi_1, \psi_2)\,, \tag{3.17}$$

where the metric $(\ ,\ )$ on the spinor fields is constructed using a trace on $y \otimes \overline{y}$ and the real structure of spinors on Y. We extend $\langle\ ,\ \rangle$ to a bilinear form on V_C by using the same formula without any complex conjugation. So $i\mathcal{D}_Y$ is antihermitean in this inner product, and hence if ψ_1 and ψ_2 are eigenvectors of $i\mathcal{D}_Y$ with eigenvalues λ_1, λ_2, then $\lambda_1 \langle \psi_1, \psi_2 \rangle = \langle i\mathcal{D}_Y \psi_1, \psi_2 \rangle = -\langle \psi_1, i\mathcal{D}_Y \psi_2 \rangle = -\lambda_2 \langle \psi_1, \psi_2 \rangle$. Consequently,

$$\langle \psi_1, \psi_2 \rangle = 0 \quad \text{unless} \quad \lambda_1 + \lambda_2 = 0\,. \tag{3.18}$$

For a generic metric on Y, the eigenvalues in (3.16) are all non zero, as we have observed above. So if we let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the subspaces of $V_{\mathbf{C}}$ generated, respectively by the eigenvectors with positive and negative eigenvalue, we have a decomposition

$$V_{\mathbf{C}} = \mathcal{S}_+ \oplus \mathcal{S}_-\,. \tag{3.19}$$

From (3.18) it follows that $\mathcal{S}_\pm$ are isotropic subspaces of V_C, that is, $\langle \psi_1, \psi_2 \rangle = 0$ for ψ_1, ψ_2 both in $\mathcal{S}_+$ or both in $\mathcal{S}_-$, and moreover, they are maximal isotropic subspaces (since a vector in $\mathcal{S}_-$, for example, is never orthogonal to its complex conjugate, which is in $\mathcal{S}_+$).

To define $\Omega(x)$, we introduce the Dirac operator $\mathcal{D}_X$ on X using Atiyah-Patodi-Singer (APS) boundary conditions [23]. This means simply that we consider the operator $\mathcal{D}_X$ to act on spinor fields on X whose restriction to Y is in $\mathcal{S}_-$. For $w \in KO(X)$, we write $j(w)$ for the mod 2 index with values in w, that is, $j(w)$ is the number mod 2 of positive chirality zero modes of $\mathcal{D}_X$, with APS boundary conditions. We define

$$\Omega(x) = (-1)^{j(x\otimes\overline{x})}\,. \tag{3.20}$$

The right-hand side is invariant under deformations of the metric on X and connection on x, as long as we only consider data for which the operator $i\mathcal{D}_Y$ has no zero eigenvalue. There is no natural extension of the definition of $\mathcal{S}_-$ when $i\mathcal{D}_Y$ develops a zero eigenvalue, so in general $\Omega(x)$ cannot be defined continuously as a $\mathbb{Z}_2$-valued function. However, we will see later that

$$\Omega(x)\,|\,\mathrm{Pf}(\mathcal{D}_Y)| \tag{3.21}$$

always varies smoothly, even when one crosses the locus on which $\mathrm{Pf}(\mathcal{D}_Y)$ develops a zero eigenvalue. This means that $\Omega(x)|\,\mathrm{Pf}(\mathcal{D}_Y)|$ can be interpreted globally as a section of $\mathrm{Pf}(Y)$, and so $\Omega(x)$ is globally a section of $\mathcal{P}(Y)$.

Given the definition of $\Omega(x)$, we can verify the gluing law in (3.15). We keep to our assumption that all boundary components, including B, have zero mod 2 index. We perform the computation using a convenient metric and gauge connection such that the Dirac operator $\mathcal{D}_B$ has no zero eigenvalues, and the metric on X looks near B like $\mathbb{R}\times B$, with this description being valid for a distance t (in the $\mathbb{R}$ direction) that is very long compared to the size of B. For $t\to\infty$, since $\mathcal{D}_B$ has no zero modes, all zero modes of $\mathcal{D}_X$ grow or decay exponentially in the $\mathbb{R}$ direction, and converge for large t to zero modes on X_1 or on X_2. Hence, the number of eigenvalues of $\mathcal{D}_X$ that converge to zero for $t\to\infty$ is the sum of the number of zero modes of $\mathcal{D}_{X_1}$ and the number of zero modes of $\mathcal{D}_{X_2}$. This implies that the mod 2 index is additive, and gives (3.15).[6]

The fact that $\Omega(x)$ is an element of $\mathcal{P}(Y)$ has implications for the quantization of the RR fields on Y in a hamiltonian framework. In quantizing the zero modes of the RR fields on Y, one gets a one-dimensional space $\mathcal{H}_y$ of quantum states for each $y\in K(Y)$. Superficially, it seems that $\mathcal{H}_y$ is canonically a copy of $\mathbf{C}$ — one describes an RR quantum state by giving a number for each y — but actually $\mathcal{H}_y$ is isomorphic to $\mathcal{P}(Y)\otimes_{\mathbb{Z}_2}\mathbf{C}$. This follows from our formalism. Since the restriction of $x\in K(X)$ to its boundary values $y\in K(Y)$ consists of mutually commuting observables, the boundary values can be simultaneously specified and treated classically. In doing so, the factor $\Omega(x)$ is a factor in the path integral, and since it takes values in $\mathcal{P}(Y)$, the path integral with boundary values y is not a number but an element of $\mathcal{P}(Y)\otimes_{\mathbb{Z}_2}\mathbf{C}$

[6] If B has a non-zero mod 2 index, $\mathcal{D}_X$ can have a zero mode that is not localized on either side. More care is then needed both here and in the definition of $\Omega(x)$.

(where again, $\mathcal{P}(Y)$ depends on y). Since the path integral on a manifold with boundary should define a quantum state in the Hilbert space of the boundary, the space of quantum ground states for given y must be isomorphic to this.

Verification of the main claim. Finally, we must show that (3.21) varies smoothly as the data on Y vary. We will explain this by analogy with a finite dimensional situation. Let V be a real vector space of dimension $2\,k$ for some k with a positive definite inner product which we denote $\langle v, w\rangle$ for $v, w \in V$. Let $V_{\mathbf{C}}$ be the complexification of V, to which we extend $\langle\ ,\ \rangle$ as a bilinear form, and let $\mathcal{S}$, $\mathcal{S}'$ be maximal isotropic subspaces of $V_{\mathbf{C}}$, that is, k-dimensional subspaces such that $\langle v, w\rangle = 0$ for $v, w \in \mathcal{S}$ or $v, w \in \mathcal{S}'$. Then the dimension of the intersection $\mathcal{S} \cap \mathcal{S}'$ is invariant mod 2 under deformations of $\mathcal{S}$ and $\mathcal{S}'$ (as maximal isotropic subspaces). This can be proved using the one-dimensional Dirac operator on the unit interval $I = [0, 1]$ for a fermi field χ with values in V. In one dimension, the spin representation is one-dimensional, so the total number of components of χ is the dimension $2\,k$ of V. There is no room for curvature or holonomy, so we can take the Dirac operator on the interval to be just $\mathcal{D}_I = d/dt$, $0 \leq t \leq 1$. We impose boundary conditions that $\chi(0) \in \mathcal{S}$ and $\chi(1) \in \mathcal{S}'$. The Dirac operator with these boundary conditions is elliptic and skew-symmetric (here one uses that $\mathcal{S}$ and $\mathcal{S}'$ are maximal isotropic), so it has a mod two index. A zero mode is a t-independent fermion with values in the intersection $\mathcal{S} \cap \mathcal{S}'$, so the mod 2 index equals the dimension of this intersection mod 2, and hence this dimension is a topological invariant mod 2.

There actually are two connected families of maximal isotropic subspaces. Once we pick an orientation of V, they can be described as follows. Upon picking a basis $s_1, s_2, \ldots, s_k$ for $\mathcal{S}$, we consider $\mathcal{S}$ to be self-dual or anti-self-dual depending on whether the k-form $ds_1 \wedge ds_2 \wedge \cdots \wedge ds_k$ is self-dual or anti-self-dual. If $\mathcal{S}$ and $\mathcal{S}'$ are both self-dual or both anti-selfdual, the intersection dimension is k mod 2 (this is clear upon taking $\mathcal{S} = \mathcal{S}'$), and if they are of opposite types, the intersection dimension is $k-1$ mod 2. To verify the last statement, consider the following modification of $\mathcal{S}$. We can pick a basis such that $\overline{s}_1$ (as well as s_1) is orthogonal to $s_2, s_3, \ldots, s_k$, and let $\mathcal{S}'$ be the maximal isotropic subspace spanned by $\overline{s}_1$ and $s_2, s_3, \ldots, s_k$. Then $\mathcal{S}'$ has opposite type from $\mathcal{S}$ and its intersection with $\mathcal{S}$ has dimension $k-1$. If two maximal isotropic subspaces differ in this way (by complex conjugating one basis vector), we say they differ by an elementary modification.

We will apply this formalism in an infinite dimensional case in which V is the space of all spinor fields on Y with values in $y \otimes \overline{y}$. The perturbations we consider are sufficiently soft so that the above concepts can be applied even though V is infinite dimensional. (One apparently cannot make sense of whether k is even or odd, however.) We take as above $\mathcal{S}_-$ to be the maximal isotropic subspace of $V_{\mathbf{C}}$ consisting of negative eigenvalues of $i\mathcal{D}_Y$. We let $\mathcal{W}$ be the subspace of $V_{\mathbf{C}}$ consisting of boundary values of solutions of the Dirac equation $\mathcal{D}_X\psi = 0$ for ψ a spinor field on X valued in $x \otimes \overline{x}$. $\mathcal{W}$ is isotropic since for two solutions ψ_1, ψ_2 of the Dirac equation on X (we use the same name for the restriction to Y) we have

$$\begin{aligned}\langle\psi_1,\psi_2\rangle &= \int_Y d^9\,x\,\sqrt{g}\,(\psi_1,\psi_2) = \int_X d^{10}x\sqrt{g}\,\partial_i\left(\psi_1,\Gamma^i\psi_2\right) = \\ &= \int_X d^{10}\,x\,\sqrt{g}\Big((\mathcal{D}_X\psi_1,\psi_2)+(\psi_1,\mathcal{D}_X\psi_2)\Big) = 0\,. \end{aligned} \tag{3.22}$$

General considerations about elliptic operators show that $\mathcal{W}$ is maximal isotropic. With APS boundary conditions, the space of zero modes of $\mathcal{D}_X$ is the intersection $\mathcal{W}\cap S_-$, so the mod 2 index is the dimension of this intersection mod 2.

Now suppose that by varying the metric or connection on Y, we pass through a locus L in field space on which eigenvalues of $\mathcal{D}_Y$ pass through zero. Generically, this is where $\mathcal{D}_Y$ develops zero eigenvalues. The number of such eigenvalues will be even (since the dimension of the null space of $\mathcal{D}_Y$ is a topological invariant mod 2) and generically will be precisely 2. Let ψ_1 and ψ_2 be the two-zero modes at some point on L. Near L, restricted to this two-dimensional space, $\mathcal{D}_Y$ (being a real, anti-symmetric matrix) looks like

$$\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\,, \tag{3.23}$$

where generically ϵ has a simple zero on L and changes sign as one crosses L. The linear combination of ψ_1 and ψ_2 that is in $\mathcal{S}_-$ is $\psi_1+i\psi_2$ or $\psi_1-i\psi_2$ depending on the sign of ϵ. Hence, $\mathcal{S}_-$ undergoes an elementary modification in crossing L. It follows that the mod 2 index $j(x\otimes\overline{x})$ changes by 1 in crossing L, so $\Omega(x)=(-1)^j$ changes sign in this crossing. The Pfaffian $\mathrm{Pf}(\mathcal{D}_Y)$ is proportional to ϵ, so its absolute value $|\,\mathrm{Pf}(\mathcal{D}_Y)|$ is proportional to $|\epsilon|$ and is not smooth on L. But $\Omega(x)|\,\mathrm{Pf}(\mathcal{D}_Y)|\sim \mathrm{Pf}(\mathcal{D}_Y)$ varies smoothly, as we wished to show, since $\Omega(x)$ changes sign precisely where $\mathrm{Pf}(\mathcal{D}_Y)$ does.

4. Application to RR periods in type IIA

We will now explain an illuminating application of this formalism which gave, in fact, the original motivation for writing the present paper.

Let W be the worldvolume of a D-brane in type-IIA superstring theory on a ten-dimensional spin manifold X. Its dimension is of the form $2\,k-1$ for some k. Let N_W be the normal bundle to W. W is not necessarily spin (it should be Spin^c [19]). In any event, because X is spin, letting $S(W)$ denote the spin bundle of W and $S(N_W)$ denote the spin bundle of N_W, the tensor product $S(W)\otimes S(N_W)$ exists as an ordinary vector bundle. The worldvolume fermions on W takes values in this bundle. The Dirac operator $\mathcal{D}_W$ is real or pseudoreal depending on k, and in any event, the fermion path integral, which is most naturally understood as a Pfaffian $\mathrm{Pf}(\mathcal{D}_W)$, is real. There is no problem in defining (with zeta function regularization, for example) the absolute value $|\,\mathrm{Pf}(\mathcal{D}_W)|$, but the sign may have an anomaly. Mathematically, $\mathrm{Pf}(\mathcal{D}_W)$ is naturally defined as a section of a real line bundle $\mathrm{Pf}(W)$ (the "Pfaffian line bundle") over the appropriate space of fields. This line bundle may be non trivial.

The anomaly in the fermion path integral means concretely the following. If we go around a loop in the space of W's, $\mathrm{Pf}(\mathcal{D}_W)$ might come back with the opposite sign. (In varying W, one may also vary other data such as the metric on X. To keep the notation simple, we will consider a loop of W's in a fixed X. One can also let the Chan-Paton gauge fields on W vary as one varies W, but we will omit this from the notation.)

When one goes around a loop in the space of W's, W sweeps out, if things are generic enough, a $2\,k$-dimensional submanifold $U \subset X$. To keep things simple, we will assume that this is so. In going around the loop, $\mathrm{Pf}(D)$ changes by

$$\mathrm{Pf}(D) \longrightarrow (-1)^{\nu(U)}\,\mathrm{Pf}(D)\,, \tag{4.1}$$

where, depending on the value of k, $\nu(U)$ is the ordinary or $\bmod 2$ index of the Dirac operator $\mathcal{D}_W$. We give some details below on the proof of (4.1) and the precise index theory formula for $\nu(U)$.

The sign factor in (4.1) is the global anomaly, and when it is non zero, it must be canceled by the coupling of the brane to the RR fields. The relevant factor in the path integral (from [9, 10, 11, 5]) is

$$\exp\left(i\int_W C\wedge\sqrt{\frac{\widehat{A}(W)}{\widehat{A}(N)}}\;\mathrm{ch}\,(x)\right), \tag{4.2}$$

with x the K-theory class of the gauge bundle on the brane. In going around a loop it changes by a factor

$$\exp\left(i\int_U G\wedge\sqrt{\frac{\widehat{A}(U)}{\widehat{A}(N)}}\;\mathrm{ch}\,(x)\right). \tag{4.3}$$

Thus, the condition that the argument of the path integral is single-valued in going around a loop is that

$$(-1)^{\nu(U)}\exp\left(i\int_U G\wedge\sqrt{\frac{\widehat{A}(U)}{\widehat{A}(N)}}\right)=1 \tag{4.4}$$

whenever U is the total space of a one-parameter family of brane world-volumes.

Note that a one-parameter family of W's has topology $U = W\times S^1$. But our discussion below will show essentially that (4.4) is valid whenever a D-brane can be wrapped on U. Global anomalies for a one-parameter family of p-dimensional objects often give a topological restriction that is valid in the physical problem for a wider class of $(p+1)$-manifolds than one can get, strictly, from a one-parameter family. This is such a case, and indeed we will consider below an example with $U = S^{2k}$. Sometimes the extension beyond what one learns directly from global anomalies can be proved using conditions of locality. Here we will get the more general result by implementing the formalism of section 3.

To try to keep things simple, we will consider a case in which only the highest dimension RR field G_{2k} contributes (for example, because the lower G's are zero). The reason for considering this case is that it brings out a conceptual difficulty that we want to emphasize. The additional contributions from the lower G's make the formulas more complicated, but do not affect this conceptual difficulty.

If only G_{2k} is relevant, the condition for anomaly cancellation is that

$$(-1)^{\nu(U)} \exp\left(i \int_U G_{2k}\right) = 1\,. \tag{4.5}$$

If ν were identically 0, this would give the naively expected condition that the periods of G_{2k} are integral multiples of 2π. More generally, if there is a differential form λ in spacetime such that

$$\nu(U) = \int_U \lambda \tag{4.6}$$

for all U, then the quantization condition G_{2k} is shifted to

$$\int_U \frac{G_{2k}}{2\pi} = \frac{1}{2}\int_U \lambda + \text{integer}\,, \tag{4.7}$$

so that it is not G_{2k} but $G_{2k} + \pi\lambda$ that obeys conventional Dirac quantization. One can formulate a similar, though somewhat more abstract, statement if there exists not a differential form λ but an element $\lambda \in H^4(X;\mathbb{Z}_2)$ obeying (4.6).

The case of D2-branes was considered in [7]. (The discussion was actually carried out in M-theory and was equivalent to a type-IIA discussion with G_2 and G_0 assumed to vanish, as we have done in obtaining (4.5).) In this case, a λ with the appropriate properties exists: it is the differential form that is related to the characteristic class $p_1(X)/2$.

For a D4-brane, there is no such λ, even as an element of $H^6(X;\mathbb{Z}_2)$. (The problem also has an analog for D8-branes once one allows the Chan-Paton bundle on W to vary.) This is because there is no cohomological formula for the mod 2 index in six dimensions.[7] For any given U, one can certainly find a λ such that (4.6) is true. But there is no λ that works for all U's.

Thus, it seems impossible, or at least unreasonably complicated (involving, at best, a variety of higher order cohomology operations) to state the appropriate quantization condition for G_6 if one interprets G_6 as a differential form and states the

[7] We are grateful to D. Freed and M.J. Hopkins for explaining this to us, and to Hopkins for explaining in detail how far one can go in the direction of such a formula and what sort of topology is involved.

condition in terms of cohomology. Given the relations between RR charges and K-theory that have emerged in the last few years, one wonders if a formalism based on K-theory would make it easier to state the necessary quantization conditions. Doing this was the original goal of the present paper. For this, we have needed the explanation in section 2 of how the RR fields are classified by K-theory, and the material that we have surveyed in section 3 concerning the interpretation of self-duality in the K-theory language.

An illustrative example. We will now make explicit what the formalism of sections 2 and 3 means as applied to an illustrative example, and show that (4.5) is a consequence. (An extension of the same reasoning shows that if suitably interpreted, (4.4) is a consequence of the formalism in sections 2 and 3.)

We want to consider a simple example in which spacetime contains a $2\,k$-sphere U with normal bundle N. N is a real vector bundle of rank $10-2\,k$. For suitable choice of N, $\nu(U)$ will be non zero; we want to explore the quantization of the RR form G_{2k} on U.

We could take the spacetime manifold X to be the total space of the bundle N. However, the framework of section 3 is most straightforward for compact X. Hence, we replace N by a sphere bundle. This is done by adding a point at infinity to each fiber of $N \to U$. The fibers are copies of $\mathbb{R}^{10-2k}$; compactifying each fiber by adding a point at infinity, we compactify the total space of N to a manifold X that is a sphere bundle over U, with fibers S^{10-2k}.

A homology basis of X is given by the following four classes: a point $p \in X$; U, embedded in X as the zero section of N; a fiber F of the sphere bundle $X \to U$; and X itself. These are all spin manifolds, so a brane wrapped on any one of them with trivial Chan-Paton bundle gives an element of $K(X)$. We denote the corresponding elements of $K(X)$ as $[p], [U], [F]$, and $[X]$. Using the Atiyah-Hirzebruch spectral sequence, it can be shown $K(X) \cong \mathbb{Z}^4$ with these four classes furnishing a basis. The bilinear form on $K(X)$ is given by $([p],[X]) = 1$, $([U],[F]) = (-1)^k$, with other components vanishing. (The factor $(-1)^k$, which will not play an important role, comes from the complex conjugation of the second factor in the definition of the anti-symmetric form (x,y) on $K(X)$.)

The $\mathbb{Z}_2$-valued function Ω is completely determined by its value for the four basis elements together with the fundamental relation (3.5). We will see that if V is any even-dimensional spin submanifold of X and $[V]$ is the corresponding K-theory class, then

$$\Omega([V]) = (-1)^{\nu(V)} . \tag{4.8}$$

From this, it follows in the case at hand that $\Omega([V])$ is $+1$ if V is p, F, or X, while $\Omega([U]) = (-1)^{\nu(U)}$ is in general non trivial. ($\nu(p)$ is zero because the Dirac index on a point, with values in an even rank bundle, is zero mod 2. $\nu(F)$ is zero because F has trivial normal bundle, and positive scalar curvature, so the relevant Dirac operator has no zero modes. Finally, $\nu(X)$ is zero because X is a fiber bundle with fibers of positive scalar curvature, so the Dirac equation has no zero modes. The statements about $\nu(F)$ and $\nu(X)$ use the fact that the Chan-Paton bundles are trivial.)

Given this, let us discuss the quantization of the RR periods. The theta function of $K(X)$ is constructed as a sum over a maximal commuting lattice Λ_1, which we take to be generated by $[F]$ and $[X]$. We take the complementary lattice Λ_2 to be generated by $[p]$ and $[U]$. The theta function is constructed as a sum over the coset $\frac{1}{2}\theta + \Lambda_1$ in $\frac{1}{2}\Lambda_1$, where $\theta \in \Lambda_1$ is such that

$$\Omega(x) = (-1)^{(\theta,x)} \tag{4.9}$$

for $x \in \Lambda_2$. In view of (4.8) and the structure of the bilinear form $(\ ,\ )$, this means that

$$\theta = [F]\,. \tag{4.10}$$

Hence the theta function is constructed as a sum over the coset

$$\frac{1}{2}[F] + \Lambda_1 \subset \frac{1}{2}\Lambda_1\,. \tag{4.11}$$

The theta function is constructed, in other words, as a sum over elements of the form

$$\Big(n + \nu(U)/2\Big)\,[F] + m[X] \tag{4.12}$$

with integers n and m.

Concretely, $[F]$ corresponds to an RR form G_{2k} (since F has codimension $2\,k$) of delta function support on F, such that

$$\int_U \frac{G_{2k}}{2\pi} = 1\,. \tag{4.13}$$

Analogously, $[X]$ corresponds to an RR form G_0 such that

$$\frac{G_0}{2\pi} = 1\,. \tag{4.14}$$

(4.12) shows that the theta function is constructed as a sum over elements of $\frac{1}{2}\Lambda_1$ that correspond to RR forms with

$$\int_U \frac{G_{2k}}{2\pi} = \frac{\nu(U)}{2} \mod \mathbb{Z}\,. \tag{4.15}$$

In this sense, the mod2 index shifts the quantization of the RR forms in the expected way.

4.1 Computation of $\nu(U)$

We still need to describe why the global anomaly $\nu(U)$ is a mod 2 index, and to show that $\Omega([U])$ is determined by the same mod 2 index.

The details of the evaluation of the global anomaly depend on the dimension of the brane worldvolume W. The two cases in which $\dim(W)$ is of the form $4n-1$ are somewhat similar, so we consider them first, followed by the two rather similar cases with $\dim(W)$ of the form $4n+1$.

W of Dimension $4n-1$. If W is three dimensional (the case already considered in [7]), then the spinors on W are pseudoreal. The hermitean dirac operator $i\mathcal{D}_W = i\,\Gamma^I D_I$ on W with values in any real bundle has an antiunitary "complex conjugation" symmetry τ, with $\tau\mathcal{D}_W = \mathcal{D}_W\tau$, and $\tau^2=-1$.[8] If ψ is an eigenfunction of $\mathcal{D}_W$, then $\tau\psi$ is an eigenfunction with the same eigenvalue; it cannot be a multiple of ψ since $\tau\psi = c\psi$ for complex c would imply, given the properties of τ, that $\bar{c}c=-1$. Hence the eigenvalues of $\mathcal{D}_W$ appear in pairs. In our case, the Dirac operator $\mathcal{D}_W$ that we want acts on spinors with values in $S(N_W)$, where N_W is the normal bundle to W, and $S(N_W)$ are the spinors of N_W. N_W has rank seven, and $S(N_W)$ is a real bundle.

Now, when we go around a loop in the space of W's, eigenvalue pairs may pass through zero. Every time this occurs, the Pfaffian $\mathrm{Pf}(\mathcal{D}_W)$ changes sign. So $\nu(U)=\Delta$, where Δ is the net number of times a pair of eigenvalues passes through zero from the positive to negative direction. A one-parameter family of Dirac operators on W glue together to make a Dirac operator on $U=S^1\times W$ (assuming the metric on W is kept fixed in the family; more generally, U is a fiber bundle over S^1 with fibers copies of W, but we will not build this into the notation). A standard argument relating spectral flow in three dimensions to Dirac zero modes in four dimensions [24] shows that the index $i(S(N_W))$ of the Dirac operator with values in $S(N_W)$ is 2Δ, so $\nu(U)=i(S(N_W))/2$. This can be more conveniently written as follows. The four-manifold U has in the string theory spacetime X a normal bundle N of rank six; one has $N_W = TS^1\oplus N$ with TS^1 the tangent bundle to S^1. As TS^1 is trivial, it follows that the spin bundle $S(N_W)$ of N_W is the same as the spin bundle $S(N)$ of N. But the description in terms of N gives a simplification; as N has even rank, its spin bundle has a chiral decomposition as $S(N)=S_+(N)\oplus S_-(N)$. The two summands are related by complex conjugation, and hence the index of the Dirac operator with values in $S_+(N)$ equals that with values in $S_-(N)$. So $i(S(N_W))/2 = i(S_+(N)) = i(S_-(N))$. The final result for $\nu(U)$ is then in this case

$$\nu(U) = i(S_+(N)) = i(S_-(N))\,. \tag{4.16}$$

[8] In a local Lorentz frame, one can take the gamma matrices to be the usual 2×2 Pauli sigma matrices, with σ_2 imaginary and the others real. τ is then σ_2 times complex conjugation; it commutes with $i\sigma^I D_I$.

If W is seven dimensional, everything is the same except the details of constructing the pseudoreal symmetry τ. One can pick seven 8×8 gamma matrices Γ^i that are imaginary and square to $+1$, so the hermitean Dirac operator $i\Gamma^I D_I$ on spinors of W is real. However, we want the Dirac operator on spinors on W with values in $S(N_W)$, and (as N_W has rank three) the spinors of N_W are pseudoreal. Because of the pseudoreality of N_W, there is again a complex conjugation symmetry τ of $i\mathcal{D}_W$ with $\tau^2 = -1$.[9] The rest of the argument is the same. The eigenvalues of $i\mathcal{D}_W$ are paired by τ, and in a one-parameter family of W's, there is again a possibility of spectral flow. So again if Δ is the net number of times an eigenvalue pair passes through zero, then the number of sign changes of $\mathrm{Pf}(\mathcal{D}_W)$ is $\nu(U) = \Delta$. And again, Δ is half the index of the Dirac operator on U with values in $S(N_W)$. Once again, letting N be the normal bundle to $U = S^1 \times W$, we have $S(N_W) = S_+(N) \oplus S_-(N)$, and the same reasoning leads again to (4.16).

W of Dimension $4n+1$. Now suppose that W is five dimensional. Then (as the normal bundle to W is of rank five and spinors of SO(5) are pseudoreal) the spinors on W and the spinors on its normal bundle N_W are both pseudoreal, so the Dirac operator $\mathcal{D}_W$ is real. This means that the eigenvalues of the hermitean operator $i\mathcal{D}_W$ occur in pairs with opposite sign: if $i\mathcal{D}_W\psi = \lambda\psi$, then $i\mathcal{D}_W\overline{\psi} = -\lambda\overline{\psi}$. It is still true that the Pfaffian $\mathrm{Pf}(\mathcal{D}_W)$ changes sign every time an eigenvalue pair crosses zero, but this time the eigenvalues in the pair are crossing from opposite directions. We let Δ be the number of eigenvalue pair crossings mod 2, so $\nu(U) = \Delta$. (In contrast to the case where W has dimension $4n-1$, the number of eigenvalue pair crossings is only a topological invariant mod 2 since eigenvalues are crossing in opposite directions.)

The relation between a one parameter family of Dirac operators on W and a Dirac operator on $U = S^1 \times W$ is now that the mod 2 spectral flow Δ on W, for spinors with values in $S(N_W)$, equals the mod two index of the Dirac operator on $U = S^1 \times W$, with values in the same bundle. We recall that this mod 2 index $j(S(N_W))$ is defined as the number of positive chirality zero modes of the Dirac operator on U with values in N_W (regarded as a bundle on U), mod 2. From $N_W = TS^1 \oplus N$, we again have $S(N_W) = S(N) = S_+(N) \oplus S_-(N)$, so $j(S(N_W)) = j(S_+(N)) + j(S_-(N))$. But the two terms on the right are in general not equal, unlike the case when W has dimension $4n-1$. So our result is now

$$\nu(U) = j(S(N)) = j(S_+(N)) + j(S_-(N))\,. \tag{4.17}$$

For W nine dimensional, the analysis is much the same. One change is the explanation of why $\mathcal{D}_W$ is a real operator. SO(9) and SO(1) both have real spin representations, so the spinors of W and of its normal bundle are both real, and hence $\mathcal{D}_W$ is real.

[9] If the generators of the SU(2) structure group of N_W are Pauli matrices σ_i with σ_2 imaginary and the others real, then τ is the product of complex conjugation with σ_2.

The other change is that, since SO(1) is trivial, the spinors of the normal bundle are a trivial rank one real bundle. So to get an anomaly, we must let either the Chan-Paton bundle on W or the metric on W vary. Also, if we want to think of the total space $U = S^1 \times W$ of a family of W's as a submanifold of the spacetime X (we could consider more general cases if we adopt a somewhat more abstract notation), we must for dimensional reasons take $U = X$. As a result, the normal bundle N to U in X is of rank zero, and the notation in (4.17) needs some clarification. As there are no gamma matrices, we consider the Clifford algebra of a rank zero vector space to consist only of scalars; there is only one irreducible representation, of dimension 1, so $S_+(N)$ is a trivial one-dimensional bundle and $S_-(N)$ is empty. With this interpretation, (4.17) can be justified by the same arguments, but is perhaps more clearly written as

$$\nu(X) = j\,, \tag{4.18}$$

where X is endowed with a set of space-filling branes carrying a Chan-Paton bundle with K-theory class x, and j is the mod two index of the Dirac operator $\mathcal{D}_X$ on spinors with values in $x \otimes \overline{x}$. The justification for (4.18) is the same that we gave for W of dimension five: the number of sign changes of $\mathrm{Pf}(\mathcal{D}_W)$ in a one parameter family equals the number Δ of level crossings, mod 2; and this in turn equals the mod 2 index of the Dirac operator, in this case on $U = X$.

Summary. The main difference between these various cases is that when W has dimension $4\,n-1$, the anomaly is given by an ordinary index that can be computed using a differential form. This leads to the type of description given in [7] for W of dimension three — a shifted quantization condition that can be expressed in terms of differential forms. However, for W of dimension $4\,n+1$, we run into a mod 2 index that cannot be described cohomologically. To make sense of the anomaly in these cases, the reinterpretation of the RR fields in K-theory is extremely useful.

The above formulas for $\nu(U)$ can be stated in a more unified way. For any vector bundle T over U, let $n_+(T)$ and $n_-(T)$ be the numbers of positive and negative chirality zero modes of the Dirac operator on U with values in T. Then in all cases we have

$$\nu(U) = n_+(S_+(N)) - n_+(S_-(N)) \mod 2\,. \tag{4.19}$$

For U of dimension $4n+2$, this is equivalent to (4.17), since $j(S_\pm(N)) = n_+(S_\pm(N))$ mod 2. For U of dimension $4n$, it is equivalent to (4.16), since $i(S_+(N)) = n_+(S_+(N)) - n_-(S_+(N))$, and by complex conjugation $n_-(S_+(N)) = n_+(S_-(N))$. (4.19) is a convenient expression for comparison with the computation that we are about to perform.

4.2 Computation of Ω

Finally, we want to show that for any even-dimensional spin submanifold V of X, $\Omega([V]) = (-1)^{\nu(V)}$.

Let N be the normal bundle of V and $S_\pm(N)$ the associated spin bundles. The class $x = [V]$ is the K-theory class $(S_+(N), S_-(N))$ where $S_\pm(N)$ are understood as bundles on X in the following sense. First, let R be a tubular neighborhood of V in X. Then there is a projection $\pi : R \to V$, and one pulls back S_+, S_- to bundles on R that we will denote by the same names. Then, away from the zero section of R, one has an isomorphism $T : S_+ \leftrightarrow S_-$ via the usual tachyon field $T = \vec{\gamma} \cdot \vec{\phi}$, where $\vec{\gamma}$ are gamma matrices on $S_+(N) \oplus S_-(N)$ (as usual, the gamma matrices reverse chirality and exchange the two factors), and $\vec{\phi}$ are coordinates in the normal direction. Using this isomorphism, $x = (S_+, S_-)$ can be understood as a K-theory class on R that is trivial away from V and hence (maintaining this triviality away from R) can be extended over X. Concretely, after perhaps replacing $(S_+(N), S_-(N))$ by $(S_+(N) \oplus F, S_-(N) \oplus F)$ for some F, one can extend $S_\pm(N) \oplus F$ over X such that the tachyon field defined in R by $T = \vec{\gamma} \cdot \vec{\phi} \oplus 1$ extends over X and is an isomorphism away from V.

Before attempting to compute the mod 2 index with values in $x \otimes \overline{x}$, we consider a slightly simpler problem. Suppose that we want to compute the index of the Dirac operator on X with values in the K-theory class x. The result we want to justify is that the Dirac operator on X, for spinors with values in x, has the same index as the Dirac operator on V for spinors with values in a trivial line bundle. One way to do the computation is to consider the Dirac operator $i\mathcal{D}_X$ on X, acting on $S_+(N) \oplus S_-(N)$, or possibly $(S_+(N) \oplus F) \oplus (S_-(N) \oplus F)$. (In the computation we are about to perform, F is irrelevant, as we will see momentarily). We perturb this operator to $i\widetilde{D}_X = i\mathcal{D}_X + wT$, where T, which exchanges $(S_+(N) \oplus F)$ with $(S_-(N) \oplus F)$, is the tachyon field constructed in the last paragraph, and w is a real number that varies from 0 to infinity. For $w = 0$, $i\widetilde{D}_X = i\mathcal{D}_X$, and for $w \to \infty$, the fermions are everywhere very massive, except near V, where the mass term of the fermions with values in $S_\pm(N)$ (but not those with values in F) vanishes. Hence, eigenstates of $i\widetilde{D}_X$ whose eigenvalue is small for $w \to \infty$ are localized near N, and are sections of $S_+(N) \oplus S_-(N)$ — the details of the choice of F and the extension of the bundles over X are irrelevant. The eigenvalue problem $i\widetilde{D}_X \Psi = \lambda \Psi$ is solved, for large w and small λ, by a kind of Born-Oppenheimer approximation. First one solves the Dirac equation in the normal directions. This equation has a unique zero mode Ψ_0 — this is the basic local fact used in building p-branes as bound states of $(p+2k)$-branes for arbitrary k [13, 14]. Then one solves the Dirac equation on V with an ansatz $\Psi = \Psi_V \otimes \Psi_0$, with Ψ_V being a spinor field on V. Then Ψ_V obeys an ordinary Dirac equation on V (with values in a trivial bundle as the line bundle generated by the Ψ_0's is trivial). So the low-lying spectrum of $i\widetilde{D}_X$ converges, for $w \to \infty$, to the spectrum of $i\mathcal{D}_V$, and the two operators have the same index.

For our present purposes, we want not the ordinary index with values in x but the mod two index with values in $x \otimes \overline{x}$. First of all, with $x = (S_+(N), S_-(N))$, $\overline{x}$ is equal to either x or $-x$ depending on whether complex conjugation maps $S_\pm(N)$

to themselves or exchanges them. This depends on the rank of the normal bundle. In any event, the minus sign will not affect the mod 2 index. Now we need to compute the number of positive chirality zero modes mod 2 of the Dirac operator $i\mathcal{D}_X$ acting on $x \otimes \overline{x}$. For this, we again introduce a tachyon perturbation, and replace $i\mathcal{D}_X$ by $i\mathcal{D}_X + wT$. It is up to us to pick a convenient T, and we pick T to be the same tachyon field used in the last paragraph, acting on x alone – we do not include any tachyon field acting on $\overline{x}$. The localization argument for large w replaces the factor of x in $x \otimes \overline{x}$ with the unique zero mode Ψ_0 in the normal direction. So finally we reduce to a Dirac operator $i\mathcal{D}_V$ on V with values in $\overline{x} = \pm(S_+(N), S_-(N))$. Hence the number of zero eigenvalues of $i\mathcal{D}_X$ on spinors with values in $x \otimes \overline{x}$ is $n_+(S_+(N)) - n_+(S_-(N)) \bmod 2$. Comparing with (4.19), we see that this statement is equivalent to the claim (4.8) that we have aimed to justify.

4.3 Some remarks on D-brane global anomalies

We have thus seen that by classifying RR fields by K-theory and properly interpreting self-duality in the quantum theory, we get, without looking at D-branes, the results that would be expected to follow from type-IIA D-brane global anomalies. It would be nice also to look at the D-branes and show that their global world-volume anomalies cancel. To do this effectively, one would like to have a natural way to describe couplings of D-branes to RR fields in the K-theory language. Not having this, we will content ourselves with looking at a special case.

Before analyzing the special case, we will try to describe its theoretical significance. On a closed manifold X, the total D-brane charge must vanish (assuming the torsion anomaly described in section 3 does not come into play), so if we had a formalism in which it was manifest that the anomalies depend only on the total D-brane charge, there would be nothing to prove: the D-branes could not contribute to the anomalies, and the discussion would reduce to the formalism we have presented.

Even if X has a non-empty boundary Y, we can use the reasoning of section 2. The total D-brane charge in type IIA is measured by the RR fields on Y, which are classified by a class in $K(Y)$. Near infinity we suppose that X looks like $\mathbb{R} \times Y$. Consider a collection of D8-branes and D$\overline{8}$-branes supported on $p \times Y$, where p is a point in Y. Suppose that in the interior of X (in the compact part of X that is bounded by $p \times Y$) the RR fields vanish. The D8-D$\overline{8}$ configuration is classified by an element $y \in K(p \times Y) = K(Y)$. In crossing the branes, the RR fields "jump" by y, so as we assume they vanish in the interior of X, they are classified at infinity by the element y of $K(Y)$.

If we had a framework in which the anomalies manifestly depend only on the K-theory classes of the branes, the above example would be "universal", as it enables us to get any desired set of RR fields at infinity, and any set of branes that produce the same fields at infinity are in the same topological class. Even though we do not have a formalism with the requisite properties, it is still instructive to examine this example.

We can simplify the discussion further. We will assume that the theory has a reasonable degree of locality so that we can "cut and paste". Using this, we can reduce to the following simple situation. We let q be a point on $\mathbb{R}$ to the "interior" of p, so that cutting X on $q \times Y$ splits it into two pieces $X_1 \cup X_2$ with the following properties. X_1 is equivalent topologically to X and contains no branes, and X_2 contains the branes. In fact, X_2 is a copy of $\mathbb{R}^+ \times Y$, and the D8-D$\overline{8}$ system is wrapped on $p \times Y$ for some $p \in \mathbb{R}^+$. q corresponds to the boundary 0 of $\mathbb{R}^+$, and the other end of $\mathbb{R}^+$ at infinity corresponds to the original boundary of X. We write Y_L and Y_R for the two ends of X_2, roughly $q \times Y$ and $\infty \times Y$. X_1 and X_2 can be glued on their common boundary $Y_L = q \times Y$ to make X. The path integral on X is a product of path integrals on X_1 and X_2 with a sum over physical states on the common boundary.

As X_1 contains no branes, the path integral on X_1 is governed by the formalism of sections 2 and 3. We gave the definition of the function Ω for a manifold with boundary at the end of section 3.

The new ingredient is X_2, which does contain branes, and has the two boundary components Y_L and Y_R. On Y_L the RR fields are trivial, and on Y_R they are controlled by the K-theory class y. According to the discussion in section 3, the RR path integral on X_2 should give not a number but a section of $\mathrm{Pf}(Y_1) \otimes \mathrm{Pf}(Y_2)$, where $\mathrm{Pf}(Y_1)$ is the Pfaffian line of the Dirac operator on Y_1 with values in the K-theory class $0 \otimes \overline{0} = 0$, and $\mathrm{Pf}(Y_2)$ is the Pfaffian line of the Dirac operator on Y_2 with values in the K-theory class $y \otimes \overline{y}$. Actually, the Pfaffian line of the Dirac operator with values in the K-theory class 0 is canonically trivial, so the path integral should take values in the Pfaffian line $\mathrm{Pf}(Y_2)$, or simply (as Y_2 is isomorphic to Y) $\mathrm{Pf}(Y)$, where as in section 3, the K-theory class $y \otimes \overline{y}$ is understood in the definition of $\mathrm{Pf}(Y)$.

What about the brane anomaly? The D8-D$\overline{8}$ system contains worldvolume fermions in the adjoint representation, that is, with values in $y \otimes \overline{y}$. So the brane path integral is a section of the very same Pfaffian line $\mathrm{Pf}(Y)$ that we have just met. All is in order. There is no need to look for any additional anomaly cancellation mechanism. And that is just as well, since, there being no cohomological formula for the global holonomy of $\mathrm{Pf}(Y)$, this would be an exceedingly difficult anomaly to cancel in a more conventional way.

Acknowledgments

Work of GM was supported in part by DOE grant DE-FG-02-92ER40704. GM also thanks the Institute for Advanced Study for hospitality and the Monell Foundation for support during the initiation of this work. Work of EW was supported in part by NSF Grant PHY-9513835 and the Caltech Discovery Fund. We are grateful to D. Freed and M.J. Hopkins for encouraging us to express everything in terms of K-theory, and to M.F. Atiyah, D. Freed, M.J. Hopkins, G. Segal, and I. Singer for detailed explanations of a variety of mathematical points. We also thank E. Diaconescu for discussions about RR fields.

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